On the persistence probability for Kac polynomials and truncated orthogonal matrices

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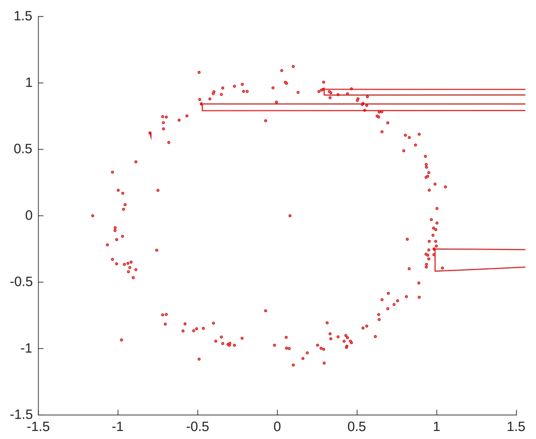
Random Polynomials

In the area of random polynomials, introduced in late 18^{th} century, one is interested in the distribution of *random* roots. There are many different models of random polynomials and here we study so called *Kac polynomials*

$$K_N(z) = \sum_{k=0}^{N} a_k z^k,$$

where $\{a_k\}_{k=0}^{\infty}$ is a family of i.i.d. mean zero, unit variance random variables having distribution ξ. It was shown before that under some mild conditions on probability distribution ξ, normalized counting measure of zeros converge to the uniform distribution on the unit circle when degree of polynomial $N \to \infty$. However for every single realization of the polynomial it is clear (see Fig. 1a) that there are many real roots.



(a) Zeros of $K_{50}(z)$ (b) Eigenvalues of M_{50}

· *How many roots are real?* Let $\mathcal{N}_N$ denote number of real roots for the polynomial K_N. Then when $N \to \infty$

$$\mathcal{N}_N \sim \mathcal{N}\left(\frac{2}{\pi}\log N, \frac{4}{\pi}\left(1 - \frac{2}{\pi}\right)\log N\right).$$

· *Is it possible to have no real roots at all?* N is assumed to be even. Heuristically, we penalize removal of every real point by multiplying the probability by $p < 1$. Thus to remove all $O(\log N)$ of them we should multiply by $p^{\log N} = N^{-\log p^{-1}}$. **Power law decay!** of persistence probability

$$p_{2n} := \mathbb{P}\{K_{2n}(x) > 0, \forall x \in \mathbb{R}\} = \frac{1}{2}\mathbb{P}\{K_{2n}(z) \text{ has no real roots}\}.$$

Random series

One can study random series instead of polynomials, i.e.

$$K(z) = \sum_{k=0}^{\infty} a_k Z^k.$$

Advantages: **Pfaffian Point Process** formed by zeros.
Disadvantages: No N left and complicated formulas.
Theorem[Matsumoto, Shirai '13] *Real zeros of Gaussian random series inside the unit disk form a PPP with the kernel*

$$K(x,y) = \begin{pmatrix} \partial_x\partial_y k(x,y) & \partial_x k(x,y) \\ \partial_y k(x,y) & k(x,y) - \alpha\varepsilon(x,y) \end{pmatrix}, \quad (1)$$

where $\alpha = -1$, $\varepsilon(x,y) = \frac{1}{2}\operatorname{sgn}(y-x)$,

$$k(x,y) = \frac{\operatorname{sgn}(y-x)}{\pi}\arcsin\frac{\sqrt{1-x^2}\sqrt{1-y^2}}{1-xy} - \varepsilon(x,y).$$

Partial analysis was announced by Fitzgerald, Tribe, Zaboronski.