Large Deviations of the Smallest Eigenvalue of Wishart-Laguerre Ensemble

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Outline

- Introduction
  - Random matrices, the Wishart ensemble
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Wishart-Laguerre Ensemble

- We consider Wishart ensemble (Biometrica, 1928)
- Distribution of the $M \times N$ matrix X is Gaussian

$$P(X) \quad \exp \quad -\frac{\beta}{2}\mathrm{Tr}X$$

Density of Eigenvalues

- From distribution of Wishart matrices joint distribution of N eigenvalues

$$ {}_N(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_0} e^{-\frac{\beta}{2}\sum_{i=1}^N \lambda_i} \prod_{i=1}^{N} \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^{\beta} $$

- First interesting object of study is the spectral density

$$ \rho_N(\lambda) = \int d\lambda_1 \cdots d\lambda_N \; {}_N(\lambda_1,\ldots,\lambda_N) \left(\frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right) $$

The Marčenko-Pastur Law

- For large N, $\rho_N(\lambda) = (1/N) f(\lambda/N)$ follows the Marčenko-Pastur law (1967)

$$f(x) = \frac{\sqrt{(\zeta_+ - x)(x - \zeta_-)}}{2\pi x} \mathbb{I}_{x \in [\zeta_-, \zeta_+]}, \quad \begin{aligned} \zeta_- &= \left(\frac{1}{\sqrt{c}} - 1\right)^2, \text{(hard edge)} \\ \zeta_+ &= \left(\frac{1}{\sqrt{c}} + 1\right)^2, \text{(soft edge)} \\ c &= \frac{N}{M} \end{aligned}$$

$c = 1/10$, $\zeta_- = 4.678$, $\zeta_+ = 17.324$

The Smallest Eigenvalue

Mathematics

- invertibility of Wishart matrix is controlled by λ_{min}
- Compressive sensing: fluctuations of λ_{min} set bounds on # of measurements to fully recover a sparse signal

Statistics

- Statistical tests based on W^{-1} (e.g. Hotelling's $T-$square test)

Physics

- Quantum information -measure of entanglement

The Smallest Eigenvalue

Exact expressions for finite N and M using various techniques, e.g.

- Edelman's approach (1991)

$$\rho_{M,N}^{(\min)}(\lambda) = C_{M,N}\lambda^{(M-N-1)/2}e^{-\lambda N/2}g_{M,N}(\lambda)$$

  with $g_{M,N}(\lambda)$ polynomials (different expressions for $M - N$ even or odd).
  These expressions (and similar ones) difficult to evaluate for large sizes.

For large N, information on the typical fluctuations of the smallest eigenvalue ($c < 1$): Tracy-Widom distribution (Feldheim & Sodin, 2010)

$$\lambda_{\min} = \zeta_- - \zeta_-^{2/3}c^{1/6}N^{-2/3}\chi_\beta\,, \quad \chi_\beta \quad TW_\beta$$

Our Goal

Study large fluctuations of the smallest eigenvalue

- simple expressions for rate functions for large deviations.

$$P_N^{(\min)}(t) \quad e^{-\beta N^2 \Phi_+^{(\min)}\left(\frac{t-N\zeta_-}{N}\right)}, \qquad N\zeta_- \leq t < \infty$$

$$P_N^{(\min)}(t) \quad e^{-\beta N \Phi_-^{(\min)}\Big(\frac{N\zeta_- - t}{}}$$

Coulomb Gas approach

- From joint distribution of eigenvalues

$$ {}_N(\boldsymbol{\lambda}) = \frac{1}{Z_0} e^{-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i} \prod_{i=1}^{N} \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^{\beta} $$

- Coulomb Gas: eigenvalues as a system of charged particles in a 2D world (logarithmic potential), constrained to the real line and external linear-log potential

$$ {}_N(\boldsymbol{\lambda}) = \frac{e^{-\beta F(\lambda)/2}}{Z_0} $$

with

$$ F(\boldsymbol{\lambda}) = \sum_{i=1}^{N} \lambda_i - \left(1 + M - N - \frac{2}{\beta}\right) \sum_{i=1}^{N} \log \lambda_i - \sum_{i=j} \log |\lambda_i - \lambda_j| $$

Coulomb Gas approach

- Quantity to calculate:

$$P_N^{(\min)}(t) \equiv \mathrm{Prob}(\lambda_{\min} \geq t) = \int_t^\infty d\lambda\, \rho_N^{(min)}(\lambda) = \frac{Z(t)}{Z_0}$$

with

$$Z(t) = \int_t^\infty \cdots \int_t^\infty e^{-\frac{\beta}{2}F(\lambda)} d\lambda_1 \cdots d\lambda_N$$

and $Z_0 = Z(t = 0)$.

- Coulomb gas with hard wall at t.

Analytics - Path integral

To obtain

$$Z(t) = \int \mathcal{D}f \, e^{-\frac{\beta}{2}N^2 S[f(x)]}$$

with

$$\begin{aligned} S[f(x)] = & \int_\zeta^\infty dx f(x) x - \left(\alpha + \frac{\beta - 2}{\beta N}\right) \int_\zeta^\infty dx f(x) \log x \\ & - \int\int_\zeta^\infty dx dy f(x) f(y) \log|x - y| \\ & + \frac{2}{\beta N} \int_\zeta^\infty dx \, f(x) \log f(x) + C_1 \left(\int_\zeta^\infty dx f(x) - 1 \right) \end{aligned}$$

with $\alpha = (1 - c)/c$.

- No Dyson correction in the entropic term

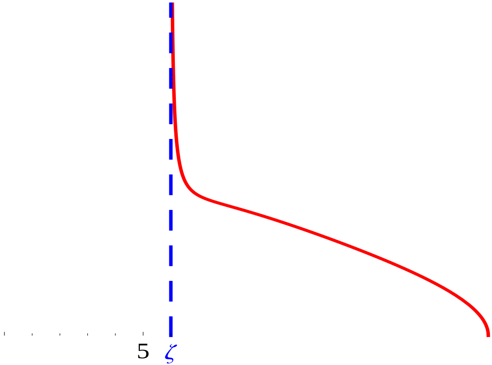
Analytics: Finite Interval Hilbert Transformation

Solution (Mathematical solution + normalisation + positivity):

An intuitive representation

An intuitive representation

An intuitive representation



An intuitive representation

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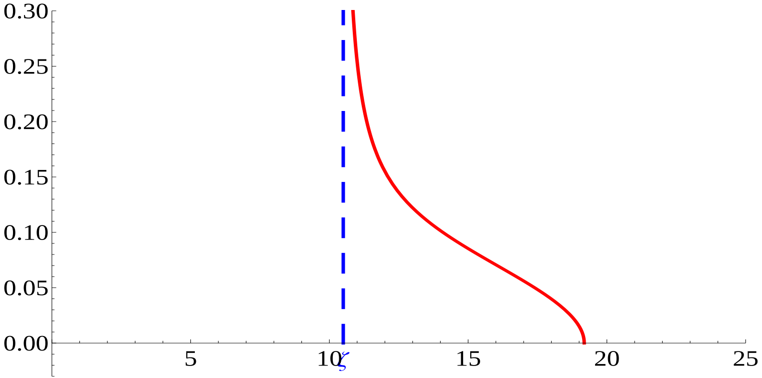


Large deviations to the left of λ_{min}

- Coulomb Gas approach (as presented) not able to capture fluctuations to the left of λ_{min}
- Reason: we only consider leading terms $O(N^2)$, which capture bulk properties

Large deviations to the left of $\lambda_{\min}$

- Energetic Argument (Majumdar & Vergassola)
- Expression the free energy $F(\boldsymbol{\lambda})$
- Energetic cost of moving the smallest eigenvalue to the left $t \quad \zeta_- N$ (this does not require a global rearrangement of the bulk)

$$\begin{aligned}
\Delta E(t) &= F(t, \lambda_2, \ldots, \lambda_N) - F(\zeta_- N, \lambda_2, \ldots, \lambda_N) \\
&= t - \alpha N \log(t) - 2 \sum_k \log|t - \lambda_k| + C \\
&= t - \alpha N \log(t) - 2N \int d\lambda\, \rho_{\mathrm{MP}}(\lambda) \log|t - \lambda| + C
\end{aligned}$$

C so that $\Delta E(t = \zeta_- N) = 0$.

Large deviations to the left of $\lambda_{\min}$

- Obtain

$$P_N^{(\min)}(t) \quad e^{-\beta N \Phi_-^{(\min)}\left(\frac{N\zeta_- - t}{N}\right)}, \qquad 0 \le t \le N\zeta_-$$

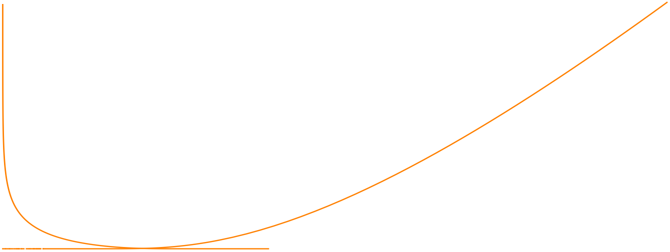
- Left rate function

$$\Phi_-^{(\min)}(x) = -\frac{\alpha}{2}\log\left(1 - \frac{x}{\zeta_-}\right) - \frac{1}{2}\sqrt{x(x+\Delta_-)} + 2\log\left(\frac{\sqrt{x+\Delta_-} - \sqrt{x}}{\sqrt{\Delta_-}}\right) + \alpha\log\left(1 + 2\frac{\sqrt{x(x+\Delta_-)} - x}{\Delta_-\sqrt{\zeta_-}}\right), \quad 0 \le x \le \zeta_-$$

with $\Delta_- = \zeta_+ - \zeta_- = 4\sqrt{1+\alpha}$.

Large deviations- Numerics

$N = 11$, $M = 110$. Comparison with Edelman's (91) for $\beta = 1$



Comparison with Tracy-Widom

$$P_N^{(\min)}(t) \equiv \lim_{N\to\infty} \mathrm{P}_{\beta,N}\left[(\lambda_{\min} - z_N^{(\beta)})/s_N^{(\beta)} \le t\right]$$

To compare with Tracy-Widom, expand rate functions:

$$\Phi_-^{(\min)}(x) \underset{x\to 0}{\approx} \frac{2}{3\zeta_- c^{1/4}} x^{3/2}, \quad \Phi_+^{(\min)}(x) \underset{x\to 0}{\approx} \frac{1}{24\zeta_-^2\sqrt{c}} x^3$$

Then

$$P_N^{(\min)}(t) \approx \begin{cases} \exp\left(-\frac{2\beta}{3}\chi^{3/2}(t)\right), & 0 \le t \le \zeta_- N \\ \exp\left(-\frac{\beta}{24}|\chi(t)|^3\right), & t > \zeta_- N \end{cases}$$

with $\chi(t) = -\frac{N\zeta_- - t}{N^{1/3}\zeta_-^{2/3}c^{1/6}}$

Almost Square Matrices

- $M = N + a, \alpha = a/N$, $a \to a(\beta) = a + (\beta - 2)/\beta$
- Look at the behaviour for $z = Nt$

$$P_N^{(\min)}(z) \approx \begin{cases} \exp\left(-\beta a \Psi_-^{(\min)}\left(\frac{4z}{a^2}\right)\right), & z \in [0, a^2/4] \\ \exp\left(-\beta a^2 \Psi_+^{(\min)}\left(\frac{4z}{a^2}\right)\right), & z \in [a^2/4, \infty) \end{cases}$$

with

$$\Psi_+^{(\min)}(x) = \frac{1}{8}\left(x - 4\sqrt{x} + 3 + \ln x\right),$$

$$\Psi_-^{(\min)}(x) = \ln \frac{1+\sqrt{1-x}}{\sqrt{x}} - \sqrt{1-x}$$

- $a(\beta) = 0$ ($a = 1, \beta = 1$ or $a = 0, \beta = 2$)

$$P_N^{(\min)}(z) = e^{-\beta z/2}$$

Almost Square Matrices

Comparison with Edelman's exact result for $\beta = 1$ ($N = 200$, a=5)

Subleading contributions

- Entropic contribution: Saddle-point equation

$$\frac{1}{2}(x - \alpha \log x) + \frac{1}{\beta N} \log f(x) + D = \int_{\zeta}^{\infty} dy\, f(y) \log |x - y|$$

Support of $f(x)$ is not compact fluctuations to the left of $\lambda_{\min}$

- Non-linear integral equation (Hammerstein type)
- Standard perturbation is hopeless
- Non-standard perturbation (boundary layer theory ?) as difficult as the original equation

Subleading contributions

Two options:

- simplest analytical approach: $y \quad R_{\text{interior}}, x \quad R_{\text{exterior}}$,
  $V(x) = \frac{1}{2}(x - \qquad) \; 9$

Subleading contributions

- Numerical solution (Abdou & Ismail 2002)