From Classical to Quantum Integrable Systems

Panagiota Adamopoulou and Clare Dunning
University of Kent

RMT 2014

The Ordinary Di erential Equation / Integrable Model Correspondence

The

eigenvalues

of certain ordinary di erential equations

are the Bethe roots of certain massless quantum integrable models

IM

ODE

Main result

The
eigenvalues
of linear systems of certain partial di erential equations
are the Bethe roots of certain massive quantum integrable models

IM
ODE

Lukyanov, Zamolodchikov; Dorey, Faldella, Negro, Tateo;
Adamopoulou, TCD; Ito, Locke

Ordinary Di erential Equations

The eigenvalues fE_jg of

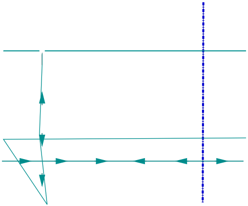
$$\frac{d^2}{dx^2} + (x^{2M} \quad E) + \frac{l(l+1)}{x^2} \qquad (x;E;l) = 0 \qquad 2\; L^2(\mathrm{R}^+)$$

satisfy Bethe ansatz equations

$$\underset{k=1}{\overset{1}{\text{¥}}} \frac{E_k \quad !^2 E_j}{E_k \quad ! \quad {}^2E_j} = \quad e^{\frac{i \quad (2l+1)}{M+1}} \qquad j = 1;2;:::;1$$

where $! \;= e^{\frac{i}{M+1}}$

6-vertex model



The same Bethe ansatz equations encode the ground state of the 6-vertex model with twisted boundary conditions in the thermodynamic limit

Spectral parameter		E
Anisotropy	$=$	$\frac{}{2}\frac{M}{M+1}$
Twist	$=$	$\frac{(2l+1)}{2M+2}$

XXZ model

The same Bethe ansatz equations encode the ground state of the XXZ model

$$H_{XXZ} = \frac{1}{2}\sum_{j=1}^{N} {}^{x}_{j}\;{}^{x}_{j+1} + {}^{y}_{j}\;{}^{y}_{j+1} \quad \cos 2 \quad {}^{z}_{j}\;{}^{z}_{j+1}$$

with twisted boundary conditions

$${}^{z}_{N+1} = {}^{z}_{1} \;; \qquad {}^{x}_{N+1} \quad i \; {}^{y}_{N+1} = e^{\;i2} \quad {}^{x}_{1} \quad i \; {}^{y}_{1}$$

in the thermodynamic limit

Massless quantum eld theory

The same Bethe ansatz equations encode the primary elds of a $c \quad 1$ conformal eld theory or equivalently the massless limit of the quantum sine-Gordon model

Central charge	$c(M)$	$=$	$1 \quad \frac{6M^2}{M+1}$
Highest weight	$(M;l)$	$=$	$\frac{(2l+1)^2 \quad 4M^2}{16(M+1)}$
Vacuum parameter	p	$=$	$\frac{2l+1}{4M+4}$

Massive sine-Gordon model

Massive model is de ned on a cylinder with radius s

A speci c solution of the modi ed sinh-Gordon equation

Require a unique, real solution of

$$\partial_z \partial_z \quad (z;z) \quad e^{2 \; (z;z)} + p(z)p(z)\, e^{\;2 \; (z;z)} = 0$$

that is nite everywhere except at   = 0 where

$z = \; e^{i} \; ; z = \; e^{\; i}$ with

- Periodicity

$$_{i} \quad ; \quad + \quad = M \quad = \; _{i}(\; ; \;)$$

- Asymptotic behaviour as $\;! \; 0$:

$$(\; ; \;) = l \ln \quad + \; _{0} + O(1)$$

- Asymptotic behaviour as $\;! \; 1$

$$(\; ; \;) = M = 2 \ln \quad + o(1)$$

The associated linear problem

is

$$\partial_z + U(z;z;\ ) \qquad = 0; \qquad \partial_z + V(z;z;\ ) \qquad = 0$$

where $\ = (\ _1;\ _2)^T$

The solution to the linear system is

$$(z;z;\ ) = \frac{^{1=2}e^{\ =2}}{e^{\ =2}\ \ ^{1=2}(\partial_z + \partial_z\ )\ )} = \frac{e^{\ =2}\ \ ^{1=2}(\partial_z + \partial_z\ )\ )}{^{1=2}e^{\ =2}}$$

Eliminating $\ _2$ we nd

$$\boxed{[\ \partial_z^2 + (\partial_z\ )^2 \quad \partial_z^2 \quad + \quad ^2p(z)] \quad (z;z) = 0}$$

Similarly

$$\boxed{[\ \partial_z^2 + (\partial_z\ )^2 \quad \partial_z^2 \quad + \quad ^2p(z)] \quad (z;z) = 0}$$

Classical integrability

The modi ed a ne Toda equations arrise from the zero-curvature condition $V_z \quad U_z + [U; V] = 0$ of the linear problem

$$@_z + U(z; z; \ ) \qquad = 0; \qquad @_z + V(z; z; \ ) \qquad = 0$$

with Lax matrices

$$(U(z; z; \ ))_{ij} = @_z \ {}_i \ {}_{ij} + \quad (C(z))_{ij}$$

and

$$(V(z; z; \ ))_{ij} = \quad @_z \ {}_i \ {}_{ij} + \quad {}^1(C(z))_{ji}$$

$$(C(z))_{ij} = \quad \begin{matrix} \exp(\ {}_{j+1} \quad {}_j) \ {}_{i \ 1;j} & ; \quad j = 1; \dots; n \quad 1 \\ p(z) \exp(\ {}_{j+1} \quad {}_j) \ {}_{i \ 1;j} & ; \quad j = n; \end{matrix}$$

where

$$ {}_{ij} = \begin{cases} 1 & \text{if } i \quad j \quad \text{mod } n \\ 0 & \text{otherwise} \end{cases}$$

Associated linear problem

The linear problem

$$\partial_z + U(z;\bar z;\ ) \quad = 0; \qquad \partial_{\bar z} + V(z;\bar z;\ ) \quad = 0$$

has solutions

$$_i(z;\bar z;\ ) \quad = \quad \frac{{}^{1}e^{\ i} \quad {}^{i+1}\ \partial_z \quad {}_{i+1} + \partial_z \quad {}_{i+1} \quad {}_{i+1}}{e^{\ n}}$$

$$= \quad \frac{e \quad {}^{1}}{e^{\ i \quad 1} \quad {}^{i}\ \partial_{\bar z} \quad {}_{i \quad 1} \quad \partial_{\bar z} \quad {}_{i \quad 1} \quad {}_{i \quad 1}}$$

Eliminating $\ _1;\dots;\ _{n \quad 1}$ or $\ _2;\dots;\ _n$; we obtain

$$(\ \ 1)^{n+1}D_n(\ ) + \ \ ^n p(z) \quad = \quad 0;$$

$$(\ \ 1)^{n+1}\bar D_n(\ ) + \ \ ^n \bar p(\bar z) \quad = \quad 0$$

with n^{th}-order di erential operators

$$D_n(\ ) \quad = \quad (\partial_z + 2\,\partial_z\ \ _1)(\partial_z + 2\,\partial_z\ \ _2) \quad (\partial_z + 2\,\partial_z\ \ _n);$$

$$\bar D_n(\ ) \quad = \quad (\partial_{\bar z} \quad 2\,\partial_{\bar z}\ \ _n) \quad (\partial_{\bar z} \quad 2\,\partial_{\bar z}\ \ _2)(\partial_{\bar z} \quad 2\,\partial_{\bar z}\ \ _1)$$

Massive to massless

In the *massless* limit

$$z \to 0 \;, \quad z \quad s \to 0 \;, \quad \ln \quad \to \infty$$

with

$$x = z\, e^{\frac{}{M+1}} ;\; x = z\, e^{\frac{}{M+1}} \;; \quad E = s^{nM} e^{\frac{n\ M}{M+1}}$$

yields

$$\boxed{(-1)^{n+1} D_n(\mathbf{g}) + p(x;E) \quad (x;E) = 0 \,; \quad p(x;E) = x^{nM} \quad E}$$

where

$$D_n(\mathbf{g}) = \quad \partial_z \quad \frac{g_{n-1} \quad (n-1)}{x} \quad \partial_z \quad \frac{g_{n-2} \quad (n-2)}{x} \quad \partial_z \quad \frac{g_0}{x}$$

This is precisely the n^{th}-order ODE appearing in the massless A_{n-1} ODE/IM correspondence