Polynomial Ensembles of Derivative Type

Holger Kösters

(joint work with Mario Kieburg)

Bielefeld University

Overview

Overview

1. Motivation

2. Polynomial Ensembles . 0.606.616 cm Q n Q 1 - 0.98032 0.98047 0.9

Motivation

Standing Assumption:

Let $\mathbf{X}_n$ be a *bi-invariant* (*bi-unitarily invariant*) (*isotropic*) random matrix with values in $G := \mathrm{GL}(n; \mathbb{C})$.

$$\mathbf{X}_n \text{ bi-invariant } :\Leftrightarrow \quad \text{for any } V, W \in K := \mathrm{U}$$

Standing Assumption:
Let $\mathbf{X}_n$ be a *bi-invariant* (*bi-unitarily invariant*) (*isotropic*) random matrix with values in $G := \mathrm{GL}(n; \mathbb{C})$.

$$\mathbf{X}_n \text{ bi-invariant } :\Leftrightarrow \quad \text{for any } V, W \in K := \mathrm{U}(n), \; V \mathbf{X}_n W \overset{d}{=} \mathbf{X}_n$$

Associated Densities:

- (bi-invariant) matrix density: $f_G(g)$ on $\mathrm{GL}(n; \mathbb{C})$
  jpdf of the matrix entries w.r.t. *Lebesgue measure* on $\mathbb{C}^{n \times n}$
- (symmetric) singular value density: $f_{SV}(a) = f_{SV}(a_1, \ldots, a_n)$ on $\mathbb{R}_+^n$
  jpdf of the *squared* singular values
- (symmetric) eigenvalue density: $f_{EV}(z) = f_{EV}(z_1, \ldots, z_n)$ on $\mathbb{C}^n$
  jpdf of the eigenvalues

Basic Question:
What can we say about these densities (and the relation between them)?

Motivation

Ginibre Matrix

$$f_{SV}(a) \propto |\Delta_n(a)|^2 \prod_{j=1}^{n} e^{-a_j}$$ Fisher (1939), Hsu (1939), Roy (1939), ...

$$f_{EV}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} e^{-|z_j|^2}$$ Ginibre (1965)

Motivation

Ginibre Matrix

$f_{SV}(a) \propto |\Delta_n(a)|^2 \prod_{j=1}^{n} e^{-a_j}$ — Fisher (1939), Hsu (1939), Roy (1939), ...

$f_{EV}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} e^{-|z_j|^2}$ — Ginibre (1965)

Product of p Ginibre Matrices

$f_{SV}(a) \propto \Delta_n(a) \det\left(\left(-a_k \frac{\partial}{\partial a_k}\right)^{j-1} w_{p,0}(a_k) \right)$ — Akemann–Kieburg–Wei (2013)

$f_{EV}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} w_{p,0}(|z_j|^2)$ — Akemann–Burda (2012)

where $w_{p,q}(x) = G^{p,q}_{q,p}\left(\begin{matrix} -n,\dots,-n \\ 0,\dots,0 \end{matrix} \middle| x \right) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma^p(s)\, \Gamma^q(1+n-s)\, x^{-s}\, ds$

Motivation

Ginibre Matrix

$$f_{\mathrm{SV}}(a) \propto |\Delta_n(a)|^2 \prod_{j=1}^{n} e^{-a_j}$$ Fisher (1939), Hsu (1939), Roy (1939), ...

$$f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} e^{-|z_j|^2}$$ Ginibre (1965)

Motivation

Common Structure:

$$f_{SV}(a) \propto \Delta_n(a) \det\left(\left(-a_k \frac{\partial}{\partial a_k}\right)^{j-1} w(a_k) \right)$$

$$f_{EV}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} w(|z_j|^2)$$

Several Examples:

- products of independent [inverse] Ginibre matrices
- products of independent [inverse] truncated unitary matrices
- mixed products

Akemann–Burda (2012), Akemann–Strahov (2013), Akemann–Kieburg–Wei (2013), Akemann–Ipsen–Kieburg (2013), Adhikari–Reddy–Reddy–Saha (2013), Ipsen–Kieburg (2014), Akemann–Burda–Kieburg–Nagao (2014), Forrester (2014), Akemann–Ipsen–Strahov (2014), Kuijlaars–Zhang (2014), Kuijlaars–Stivigny (2014), Kieburg–Kuijlaars–Stivigny (2015), Kuijlaars (2015), Akemann–Ipsen (2015), Claeys–Kuijlaars–Wang (2015), ...

Overview

Polynomial Ensembles of Derivative Type

Polynomial Ensemble Kuijlaars{Stivigny (2014)

$\mathbf{X}_n$ is from a *polynomial ensemble* if it is bi-invariant and

ﬁmﬁf

Polynomial Ensemble Kuijlaars–Stivigny (2014)

$\mathbf{X}_n$ is from a *polynomial ensemble* if it is bi-invariant and

$$f_{SV}(a) \propto \Delta_n(a) \det\left(w_j(a_k) \right)_{j,k=1,\dots,n}$$

for some weight functions $w_1, \dots, w_n$ (with suitable properties).

Abbreviation: $\mathbf{X}_n \sim PE(w_1, \dots, w_n)$.

Theorem (Transfer Law) Kuijlaars–Stivigny (2014)

If $\mathbf{X}_n \sim PE(w_1, \dots, w_n)$ and $\mathbf{Y}_n \sim$ Ginibre are independent,
then $\mathbf{X}_n \mathbf{Y}_n \sim PE(w_1 \circledast w_{Gin}, \dots, w_n \circledast w_{Gin})$, where $w_{Gin}(x) := e^{-x}$.

Kuijlaars–Stivigny (2014), Kieburg–Kuijlaars–Stivigny (2015), Kuijlaars (2015), Claeys–Kuijlaars–Wang (2015), ...

Polynomial Ensemble of Derivative Type Kieburg–K. (2016)

$\mathbf{X}_n$ is from a *polynomial ensemble of derivative type* if it is bi-invariant and

$$f_{SV}(a) \propto \Delta_n(a) \det\left(\left(-a_k \frac{\partial}{\partial a_k}\right)^{j-1} w_0(a_k) \right)_{j,k=1,\dots,n}$$

for some weight function w_0 (with suitable properties).

Abbreviation: $\mathbf{X}_n \sim DPE(w_0)$.

Polynomial Ensemble of Derivative Type Kieburg–K. (2016)

$\mathbf{X}_n$ is from a *polynomial ensemble of derivative type* if it is bi-invariant and

$$f_{SV}(a) \propto \Delta_n(a) \det\left(\left(-a_k \frac{\partial}{\partial a_k}\right)^{j-1} w_0(a_k) \right)_{j,k=1,\ldots,n}$$

for some weight function w_0 (with suitable properties).

Abbreviation: $\mathbf{X}_n \sim DPE(w_0)$.

Examples from RMT

- induced Wishart–Laguerre ensemble: $w_0(a) = a^{\nu} e^{-a}$
- induced Jacobi ensemble: $w_0(a) = a^{\nu} (1-a)^{\mu-1} \mathbf{1}_{(0,1)}(a)$
- induced Cauchy–Lorentz ensemble: $w_0(a) = a^{\nu} (1+a)^{-\mu-1}$

- products of such random matrices: $w_0(a) =$ Meijer-G-function

---

- Muttalib–Borodin ensemble (of Wishart–Laguerre type)

  Muttalib (1995), Borodin (1999), Cheliotis (2014), Forrester–Liu (2014), Forrester–Wang (2015), Zhang (2015), ...

  (a) $f_{SV}(a) \propto \Delta_n(a)\, \Delta_n(a^{\theta})\, (\det a)^{\nu}\, e^{-\operatorname{tr} a}$ $\qquad w_0(a) = a^{\nu} e^{-a}$

  (b) $f_{SV}(a) \propto \Delta_n(a)\, \Delta_n(\ln a)\, (\det a)^{\nu}\, e^{-\operatorname{tr}(\ln a)^2}$ $\qquad w_0(a) = a^{\nu} e^{-(\ln a)^2}$

Main Results

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{\mathrm{SV}}(a) \propto \Delta_n(a) \det\left(\left(-a_k \frac{\partial}{\partial a_k}\right)^{j-1} w_0(a_k) \right)_{j,k=1,\ldots,n}$.

Theorem (Eigenvalue Density) Kieburg–K. (2016)

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} w_0(|z_j|^2)$.

Main Results

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{\mathrm{SV}}(a) \propto \Delta_n(a) \det\left(\left(-a_k \frac{\partial}{\partial a_k}\right)^{j-1} w_0(a_k) \right)_{j,k=1,\ldots,n}$.

Theorem (Eigenvalue Density) Kieburg–K. (2016)

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^n w_0(|z_j|^2)$.

Theorem (Transfer Law) Kieburg–K. (2016)

Let $\mathbf{X}_n \sim DPE(w_0)$ and $\mathbf{Y}_n \sim DPE(v_0)$ be independent.

Then $\mathbf{X}_n \mathbf{Y}_n \sim DPE_n(w_0 \circledast v_0)$, where

$$(w_0 \circledast v_0)(x) = \int_0^\infty w_0(xy^{-1}) v_0(y) \frac{dy}{y}, \quad x > 0.$$

Main Results

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{\mathrm{SV}}(a) \propto \Delta_n(a) \det\left(\left(-a_k \frac{\partial}{\partial a_k}\right)^{j-1} w_0(a_k) \right)_{j,k=1,\dots,n}$.

Theorem (Eigenvalue Density) Kieburg–K. (2016)

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^n w_0(|z_j|^2)$.

Theorem (Transfer Law) Kieburg–K. (2016)

Let $\mathbf{X}_n \sim DPE(w_0)$ and $\mathbf{Y}_n \sim DPE(v_0)$ be independent.
Then $\mathbf{X}_n \mathbf{Y}_n \sim DPE_n(w_0 \circledast v_0)$, where

$$(w_0 \circledast v_0)(x) = \int_0^\infty w_0(xy^{-1}) v_0(y) \frac{dy}{y}, \quad x > 0.$$

More generally, let $\mathbf{X}_n \sim PE(w_1, \dots, w_n)$ and $\mathbf{Y}_n \sim DPE(v_0)$ be independent.
Then $\mathbf{X}_n \mathbf{Y}_n \sim PE(w_1 \circledast v_0, \dots, w_n \circledast v_0)$.

Main Results

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{\text{SV}}(a) \propto \Delta_n(a) \det\left(\left(-a_k \frac{\partial}{\partial a_k}\right)^{j-1} w_0(a_k) \right)_{j,k=1,\ldots,n}$.

Theorem (Eigenvalue Density) Kieburg–K. (2016)

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{\text{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} w_0(|z_j|^2)$.

Theorem (Transfer Law) Kieburg–K. (2016)

Let $\mathbf{X}_n \sim DPE(w_0)$ and $\mathbf{Y}_n \sim DPE(v_0)$ be independent.
Then $\mathbf{X}_n \mathbf{Y}_n \sim DPE_n(w_0 \circledast v_0)$, where

$$(w_0 \circledast v_0)(x) = \int_0^\infty w_0(xy^{-1}) v_0(y) \frac{dy}{y}, \quad x > 0.$$

More generally, let $\mathbf{X}_n \sim PE(w_1, \ldots, w_n)$ and $\mathbf{Y}_n \sim DPE(v_0)$ be independent.
Then $\mathbf{X}_n \mathbf{Y}_n \sim PE(w_1 \circledast v_0, \ldots, w_n \circledast v_0)$.

Theorem (Transfer Law) Kieburg–K. (2016)

Let $\mathbf{X}_n \sim DPE(w_0)$. Then $\mathbf{X}_n^{-1} \sim DPE(\widetilde{w}_0)$, where $\widetilde{w}_0(x) = w_0(x^{-1})\, x^{-n-1}$.

Key Tool: Spherical Transform

Univariate Situation

X : r.v. with values in $\mathbb{C}$
and a rotation-invariant density f_X

dist. of $X \longleftrightarrow$ dist. of $|X|^2$

$|X|^2$: r.v. with values in $\mathbb{R}_+$
and a density $f_{|X|^2}$

Mellin Transform

$$M_X(s) = \int_{\mathbb{R}_+} f_{|X|^2}(y)\, y^s \frac{dy}{y} = \int_{\mathbb{C}} f_X(x)\, |x|^{2s} \frac{dx}{|x|^2}$$

for suitable $s \in \mathbb{C}$

Uniqueness Theorem
$M_{X_1} = M_{X_2} \Rightarrow X_1 \stackrel{d}{=} X_2$

Multiplication Theorem
X_1, X_2 ind. $\Rightarrow M_{X_1 X_2} = M_{X_1} M_{X_2}$

Multivariate Situation

$\mathbf{X}$: r.v. with values in $GL(n;\mathbb{C})$
and a bi-invariant density $f_{\mathbf{X}}$

dist. of $\mathbf{X} \longleftrightarrow$ dist. of $\mathbf{X}^*\mathbf{X}$

$\mathbf{X}^*\mathbf{X}$: r.v. with values in $\mathrm{Pos}(n;\mathbb{C})$
and a conjugation-invariant density $f_{\mathbf{X}^*\mathbf{X}}$

Spherical Transform

$$S_{\mathbf{X}}(s) = \int_{\mathrm{Pos}(n;\mathbb{C})} f_{\mathbf{X}^*\mathbf{X}}(y)\, \varphi_s(y) \frac{dy}{(\det y)^n} = \int_{GL(n;\mathbb{C})} f_{\mathbf{X}}(x)\, \varphi_s(x^*x) \frac{dx}{|\det x|^{2n}}$$

for suitable $s \in \mathbb{C}^n$

Uniqueness Theorem
$S_{\mathbf{X}_1} = S_{\mathbf{X}_2} \Rightarrow \mathbf{X}_1 \stackrel{d}{=} \mathbf{X}_2$

Multiplication Theorem
$\mathbf{X}_1, \mathbf{X}_2$ ind. $\Rightarrow S_{\mathbf{X}_1\mathbf{X}_2} = S_{\mathbf{X}_1} S_{\mathbf{X}_2}$

Key Tool: Spherical Transform

Spherical Transform

$$S_{\mathbf{X}}(s) = \int_{GL(n;\mathbb{C})} f_{\mathbf{X}}(x)\, \varphi_s(x^* x) \frac{dx}{|\det x|^{2n}} \qquad (s \in \mathbb{C}^n)$$

Key Tool: Spherical Transform

Spherical Transform

$$S_{\mathbf{X}}(s) = \int_{GL(n,\mathbb{C})} f_{\mathbf{X}}(x)\, \varphi_s(x^* x)\, \frac{dx}{|\det x|^{2n}} \qquad (s \in \mathbb{C}^n)$$

Spherical Function

Let $x \in GL(n,\mathbb{C})$, let $\mathbf{X}_n \in GL(n,\mathbb{C})$ be a bi-invariant random matrix with the same singular values as x, and let $\mathbf{X}_n = \mathbf{Q}_n \mathbf{R}_n$ be its *QR decomposition*.

$$\varphi_s(x^* x) := \mathbb{E}\left(\prod_{j=1}^{n} \mathbf{R}_{jj}^{2(s_j + \varrho_j - 1)} \right) \qquad \left(\text{where } \varrho_j := j - \tfrac{n+1}{2}\right)$$

Key Tool: Spherical Transform

Spherical Transform

$$S_{\mathbf{X}}(s) = \int_{GL(n;\mathbb{C})} f_{\mathbf{X}}(x)\, \varphi_s(x\ \ x)\, \frac{dx}{|\det x|^{2n}} \qquad (s \in \mathbb{C}^n)$$

Spherical Function

Let $x \in GL(n;\mathbb{C})$, let $\mathbf{X}_n \in GL(n;\mathbb{C})$ be a bi-invariant random matrix with the same singular values as x, and let $\mathbf{X}_n = \mathbf{Q}_n\mathbf{R}_n$ be its *QR decomposition*.

$$\varphi_s(x\ \ x) := \mathbb{E}\left[\prod_{j=1}^{n} \mathbf{R}_{jj}^{2(s_j + \varrho_j \quad 1)}\right] \qquad (\text{where } \varrho_j := j \quad n$$

Proposition (Spherical Transform) Kieburg–K. (2016)

If $\mathbf{X}_n \sim DPE(w_0)$, then $S_{\mathbf{X}_n}(s) \propto \prod_{k=1}^{n} (\mathcal{M} w_0)(s_k - \frac{n-1}{2})$.

Proposition (Spherical Transform) Kieburg{K. (2016)

If $\mathbf{X}_n \quad DPE(w_0)$, then $S_{\mathbf{X}_n}(s) \; / \; \prod_{k=1}^{n} (\, M w_0)(s_k \quad \frac{n \; 1}{2}):$

Proof:

Set $Df(x) := (\quad x) f^{0}(x)$ and note that $M(Df)(s) = s \; M f(s)$.

$$S_{\mathbf{X}}(s) = \int_{GL(n;\mathbb{C})} f_{\mathbf{X}}(x) \; '_{s}(x \;\; x) \frac{dx}{j \det x j^{2n}} = \int_{(0;\,1\,)^n} f_{SV}(\;\;) \; '_{s}(\;\;) \frac{d}{(\det \;\;)^n}$$

$$/ \int_{(0;\,1\,)^n} {}_{n}(\;\;) \det \;\; D^{j \;\; 1} w_0(\;\; {}_k) \frac{\det \;\; (\;\; {}_j)^{s_k + (n \;\; 1)=2}}{{}_{n}(s) \;\; {}_{n}(\;\;)} \frac{d}{(\det \;\;)^n}$$

$$/ \frac{1}{{}_{n}(s)} \det \int_0^{1} D^{j \;\; 1} w_0(\;\;) \;\; ^{s_k \;\; (n+1)=2} d$$

$$= \frac{\det \;\; (s_k \quad \frac{n \; 1}{2})^{j \;\; 1} (\, M w_0)(s_k \quad \frac{n \; 1}{2})}{{}_{n}(s)} = \prod_{k=1}^{n} (\, M w_0)(s_k \quad \frac{n \; 1}{2}):$$

Theorem (Transfer Law for the Product) Kieburg–K. (2016)

Let $\mathbf{X}_n \sim DPE(w_0)$ and $\mathbf{Y}_n \sim DPE(v_0)$ be independent.
Then $\mathbf{X}_n \mathbf{Y}_n \sim DPE_n(w_0 \circledast v_0)$, where $(w_0 \circledast v_0)(x) = \int_0^\infty w_0(xy^{-1}) v_0(y) \frac{dy}{y}$.

Proof:

$$\mathbf{X}_n \sim DPE(w_0)\,; \quad \mathbf{Y}_n \sim DPE(v_0)$$

$$\Rightarrow \quad S_{\mathbf{X}_n}(s) \propto \prod_{k=1}^{n} \mathcal{M} w_0(s_k - \tfrac{n-1}{2})\,; \quad S_{\mathbf{Y}_n}(s) \propto \prod_{k=1}^{n} \mathcal{M} v_0(s_k - \tfrac{n-1}{2})$$

$$\Rightarrow \quad S_{\mathbf{X}_n \mathbf{Y}_n}(s) \propto \prod_{k=1}^{n} \mathcal{M}(w_0 \circledast v_0)(s_k - \tfrac{n-1}{2})$$

$$\Rightarrow \quad \mathbf{X}_n \mathbf{Y}_n \sim DPE(w_0 \circledast v_0)$$

Theorem (Eigenvalue Density) Kieburg–K. (2016)

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} w_0(|z_j|^2)$.

Proof:

$$f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} |z_j|^{2n-2j} \int_T f_G(zt)\,dt$$

$$f_{(\mathbf{R}_{11},\ldots,\mathbf{R}_{nn})}(r_1,\ldots,r_n) \propto \prod_{j=1}^{n} |z_j|^{2n-2j} \int_T f_G(rt)\,dt$$

Theorem (Eigenvalue Density) Kieburg–K. (2016)

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} w_0(|z_j|^2)$.

Proof:

$$f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} |z_j|^{2n-2j} \int_T f_G(zt)\,dt$$

$$f_{(\mathbf{R}_{11},\dots,\mathbf{R}_{nn})}(r_1,\dots,r_n) \propto \prod_{j=1}^{n} |z_j|^{2n-2j} \int_T f_G(rt)\,dt$$

Thus, since $S_{\mathbf{X}}$ is essentially the (componentwise) Mellin transform of $f_{(\mathbf{R}_{11},\dots,\mathbf{R}_{nn})}$, we get $f_{(\mathbf{R}_{11},\dots,\mathbf{R}_{nn})}$, and hence f_{EV}, from $S_{\mathbf{X}}$ by (componentwise) Mellin inversion:

$$S_{\mathbf{X}_n}(s) \propto \prod_{j=1}^{n} \mathcal{M}w_0(s_j - \tfrac{n-1}{2}) \iff f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} w_0(|z_j|^2)$$

Overview

Interpolation between Product Ensembles I

Product of p Ginibre Matrices and q Inverse Ginibre Matrices

$$f_{\mathrm{SV}}^{(p,q)}(a) \propto \Delta_n(a) \det\left[\left(-a_k \frac{\partial}{\partial a_k}\right)^{j-1} w_{p,q}(a_k)\right]$$

$$f_{\mathrm{EV}}^{(p,q)}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} w_{p,q}(|z_j|^2)$$

where $w_{p,q}(x) = \dfrac{1}{2\pi i} \displaystyle\int_{1-i\infty}^{1+i\infty} \Gamma^p(s)\,\Gamma^q(1+n-s)\,x^{-s}\,ds$

Interpolation between Product Ensembles I

Product of p Ginibre Matrices and q Inverse Ginibre Matrices

$$f_{\mathrm{SV}}^{(p,q)}(a) \propto \Delta_n(a) \det\left(\left(-a_k \frac{\partial}{\partial a_k}\right)^{j-1} w_{p,q}(a_k)\right)$$

$$f_{\mathrm{EV}}^{(p,q)}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} w_{p,q}(|z_j|^2)$$

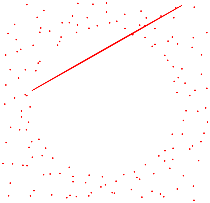
where $w_{p,q}(x) = \dfrac{1}{2\pi i} \displaystyle\int_{1-i\infty}^{1+i\infty} \Gamma^p(s)\, \Gamma^q(1+n-s)\, x^{-s}\, ds$

Problem: Is it possible to interpolate between these "product ensembles"?

Motivation: *heavy-tailed* one-point densities (in the limit as $n \to \infty$)

$$g_{\mathrm{SV}}^{(p,q)}(a) \sim a^{-\frac{q+3}{q+1}} \quad (a \to \infty)$$

$$g_{\mathrm{EV}}^{(p,q)}(z) \sim |z|^{-\frac{2q+2}{q}} \quad (z \to \infty)$$



Interpolation between Product Ensembles II

Product of p Ginibre Matrices and q Inverse Ginibre Matrices

$$f_{\rm SV}^{(p,q)}(a) \propto \Delta_n(a)\det\left[\left(-a_k\frac{\partial}{\partial a_k}\right)^{j-1} w_{p,q}(a_k)\right]$$

$$f_{\rm EV}^{(p,q)}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} w_{p,q}(|z_j|^2)$$

where $w_{p,q}(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma^p(s)\,\Gamma^q(1+n-s)\,x^{-s}\,ds$

Connection to Polya Frequency Functions

Polya Frequency Function

A function $f : \mathbb{R} \to \mathbb{R}$ is called *Polya frequency function* of order n (PF_n) if

$$\det(f(x_j - y_k))_{j,k=1,\ldots,m} \geq 0$$

for any $m = 1, \ldots, n$ and any $x_1 < \ldots < x_m$, $y_1 < \ldots < y_m$.

Connection to Polya Frequency Functions

Polya Frequency Function

A function $f : \mathbb{R} \to \mathbb{R}$ is called *Polya frequency function* of order n (PF_n) if

$$\det(f(x_j - y_k))_{j,k=1,\ldots,m} \geq 0$$

for any $m = 1, \ldots, n$ and any $x_1 < \ldots < x_m$, $y_1 < \ldots < y_m$.

Proposition Kieburg–K. (2016)

If $f \in PF_n$ (with suitable differentiability and integrability properties), then $w_0 := f \circ \log$ gives rise to a random matrix $\mathbf{X}_n \sim DPE(w_0)$.

Summary and Outlook

Polynomial Ensembles of Derivative Type

- singular value and eigenvalue densities with special *determinantal* structure
- special structure is preserved under independent products (transfer laws)
- key tool: spherical transform
- many examples (from RMT and via Polya frequency functions)

Summary and Outlook

Polynomial Ensembles of Derivative Type

- singular value and eigenvalue densities with special *determinantal* structure
- special structure is preserved under independent products (transfer laws)
- key tool: spherical transform
- many examples (from RMT and via Polya frequency functions)

Open Problems (partly work in progress)

- which functions w_0 de ne polynomial ensembles of derivative type?
- limiting spectral distributions at the *global* and *local* level?
- closer connection to free probability (re ned convergence results?)
- power-law decay: new applications of random matrix theory?
- extension to real and quaternionic matrices?