Random Fermionic Systems

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Background

- First introduced to study magnetic properties of matter
- Toy model for quantum information { study of entanglement
- Random matrix aspect

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Three papers that inspired this work:

- Lieb-Schultz-Mattis \Two soluble models of an Antiferromagnetic chain"
- Doctoral thesis of Huw Wells supervised by Jon K4.976 cm 1d bychain"

Our object of study: the Hamiltonian

- Self-adjoint operator acting on $\mathbb{C}^{2^n}$
- $$H = \frac{1}{2} \sum_{i,j=1}^{n} A_{ij}(c_i^\dagger c_j - c_i c_j^\dagger) + B_{ij}(c_i c_j - c_i^\dagger c_j^\dagger)$$

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  with $A_{ij} = A_{ji},\ B_{ij} = -B_{ij}$, i.e. $A = A^t$ and $B = -B^t$.
- c_j's are fermionic i.e. $\{c_i, c_j\} = 0, \{c_i, c_j^\dagger\} = \delta_{ij}$,

We take A_{ij}, B_{ij} iid real. Our conclusions:

- Ground state energy gap $O(1/n)$ with explicit formula if Gaussian entries
- DOS – Gaussian universally, also for A, B band
- No repulsion – numerics

Universality

- Gaussian DOS vastly universal
- Subset sums: given a set $\{\ _1, \ldots, \ _n\}$ and $S_j \quad \{1, \ldots, n\}$, eigenvalues of H are closely related to $\sum_{k \in S_j} \ _k$.
- A lot of information
  - Gaussian DOS
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- Relation to sums of weighted binomial random variables – can take Fourier transform explicitly!

Fermionic systems: how they arise?

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- nearest neighbor interaction { the XY model

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- n sites with spins that are linear combinations of ^{x} and ^{y} (no ^{z})
- nearest neighbor interaction { the XY model
- the corresponding Hamiltonian is

$$\sum_{k=1}^{n} \sum_{a \in \{x,y\}} \sum_{b \in \{x,y\}} {}_{k;a;b} \; {}^{a}_{k} \; {}^{b}_{k+1}$$

- Here ${}^{(a)}_{j} = I_2^{\;(j\ \ 1)} \quad {}^{(a)} \quad I_2^{\;(n\ \ j)}$

Jordan-Wigner transformation

- Maps a spin chain to a quadratic form in fermionic operators: allows for an exact solution
- In reverse: model a system of interacting fermions on a quantum computer

Jordan-Wigner details

- Raising and lowering operators $a_i^y = \;{}_i^x + i\;{}_i^y$ and $a_i = \;{}_i^x \quad i\;{}_i^y$
- Can recover Pauli spin operators by ${}_j^x = (a_j^y + a_j)=2$, ${}_j^y = (a_j^y \quad a_j)=2$, ${}_j^z = (a_j^y a_j \quad 1=2)$

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- Not fermionic
  - Partly fermionic: $f a_j ; a_j^y g = 1; a_j^2 = (a_j^y)^2 = 0$
  - Partly bosonic: $[a_j^y ; a_k^y] = [a_j^y ; a_k^y] = [a_j ; a_k] = 0$
- For fermionic let

$$c_j = \exp \quad i \sum_{k=1}^{j\;1} a_k^y a_k \quad a_j$$

$$c_j^y = a_j^y \exp \quad i \sum_{k=1}^{j\;1} a_k^y a_k \quad :$$

c_j's and c_j^y's are fermionic: $f c_j ; c_k^y g = \;{}_{kj}; f c_j ; c_k g = f c_j^y ; c_k^y g = 0$

Lieb-Schultz-Mattis Antiferromagnetic Chain '61

- $H = \sum_j (1 + \quad) \, {}^{x}_{j} \, {}^{x}_{j+1} + (1 \qquad) \, {}^{y}_{j} \, {}^{y}_{j+1}$
- Hamiltonian is a quadratic form in Fermi operators and can be explicitly diagonialized

Lieb-Schultz-Mattis

If $H = (\underline{c}^{y}\ \underline{c})\ M \begin{pmatrix} \underline{c} \\ \underline{c}^{y} \end{pmatrix}$; with $M = \frac{1}{2}\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ for XY model as before

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

and A, B can be explicitly diagonalized.

In the '61 paper,

- Complete set of eigenstates
- General expression for the order between any two spins involving a Green's function
- Short, intermediate, and long range order for various situations

Bipartite Entanglement

Setup: XY and XX models with a constant transversal magnetic eld
Study: Entropy E_p of entanglement between subsystems

- Vidal et al. computed E_p numerically
- Jin and Korepin compute E_p for XX model using the Fisher-Hartwig conjecture, which gives the leading order asymptotics of determinants of certain Toeplitz matrices
- Keating and Mezzadri study asymptotics of entanglement of formation of ground state using RMT methods

Wells PhD thesis

Hamiltonians of the form

$$H_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \sum_{a=1}^{3} \sum_{b=1}^{3} {}_{a;b;j} \; {}_{j}^{(a)} \; {}_{j+1}^{(b)} \qquad (1)$$

for any ${}_{a;b;j} \in \mathbb{R}$ random Gaussian (some universality possible)

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Remarks:

Wells Numerics in the XY case

For a Hamiltonian of the form

$$H_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \sum_{a=1}^{3} \sum_{b=1}^{3} \alpha_{a;b;j} \, \sigma_j^{(a)} \sigma_{j+1}^{(b)} \qquad (2)$$

- Eigenvalue repulsion in the full model and lack of repulsion in the random XY model
- Convergence to a Gaussian in the random XY model
- Numerical estimate of the error in the random XY model is on the order of $1/n$ where n is the number of cubits

Extension by Erd•s and Schr•der

- Arbitrary graphs with maximal degree   total number of edges
  - Gaussian DoS
- *p*-uniform hypergraphs
  - Correspond to *p*-spin glass Hamiltonians acting on *n* distinguishable spin-1/2 particles
  - At $p = n^{1=2}$, phase transition between the normal and the semicircle
  - **quantum-classical** transition

Summary

Known:

- DoS, spectral gap in (deterministic) XY model
- DoS in a random neighbor-to-neighbor Hamiltonian with XYZ

Numerics:

- DoS in a random XY model
- Rate of convergence in the random XY model
- Lack of repulsion

We establish:

- DoS in general bilinear forms of fermionic operators
- spectral gap in special cases

Diagonalizing M

- Eigenvalue equation: $\frac{1}{2} \begin{matrix} A & \quad B \\ B & \quad A \end{matrix} \qquad \begin{matrix} _1 \\ _2 \end{matrix} = \qquad \begin{matrix} _1 \\ _2 \end{matrix} :$
- Equivalent to: $\begin{cases} A \ _1 \quad B \ _2 = 2 \quad _1; \\ B \ _1 \quad A \ _2 = 2 \quad _2: \end{cases}$
- If $\ _1 = \ _1 \quad \ _2$ and $\ _2 = \ _1 + \ _2$, then $\begin{cases} (A + B) \ _1 = 2 \quad _2; \\ (A \quad B) \ _2 = 2 \quad _1: \end{cases}$

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$$\frac{1}{4}(A + B)^T(A + B)\ {}_1 = \ ^2\ {}_1.$$

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$$\boxed{2 \quad (M) \quad (\;) \quad \sqrt{\;{}^2} \text{ is singular value of } \frac{A + B}{2}.}$$

Need Hermiticity to get new Fermi operators

- Let U be the orthogonal matrix that diagonalizes M.
- Then U is a linear canonical transformation in the sense that

$$U = \begin{matrix} G & K \\ G^T & K^T \end{matrix} \qquad \begin{cases} GG^T + KK^T = I_n \\ GK^T + KG^T = 0_n; \end{cases} \tag{3}$$

and

$$UMU^T = \frac{1}{2} \begin{matrix} & 0 \\ 0 & \end{matrix} ;$$

with = diag($_1; \dots ;$ $_n$), $_i$ 0.

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with $= \mathrm{diag}(_1; : : : ; _n)$, $_i \quad 0$.

- Let $_k; \; _k^y$ operators de ned by

—

Diagonalizing H: Fermi basis

- $_j$ acts as a lowering operator for $^y_j \; _j$ i.e. if $^y_j \; _j j \;\; i = j \;\; i$ then $^y_j \; _j \; _j j \;\; i = 0$
- ^y_j acts as a raising operator for $^y_j \; _j$

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- $^y_j\ _j$'s commute so there exists a state $j\ \ i$ which is a simultaneous eigenstate
- By raising and lowering the state $j\ \ i$ in all possible combinations, can construct a set of 2^n orthonormal states which are simultaneous eigenstates of the $^y_j\ _j$

Diagonalizing H : subset sums

The spectrum of H is characterized as follows:

$$\boxed{x \in \sigma(H) \iff \exists S \subseteq \{1, \ldots, n\} \text{ such that } x = c + \sum_{k \in S} \lambda_k .} \qquad (4)$$

where $c = -\frac{1}{2} \sum_{k=1}^{n} \lambda_k$

Ground state energy gap: important physical quantity, re ects how sensitive is the system to perturbations

Theorem 1

For A, B

Ground state energy gap: important physical quantity, re ects how sensitive is the system to perturbations

Theorem 1

For A, B with iid Gaussian entries up to symmetry, the rescaled energy gap $\sqrt{2n/\ }$ *converges in distribution to a random variable whose probability density function is*

$$f(x) = (1 + x)e^{\ \frac{x^2}{2}\ \ x}.$$

- $x_{2^n} = \sum_{j=1}^{n}\ {}_j$ and $x_{2^n\ 1} = \sum_{j=2}^{n}\ {}_j$ yielding that

$$:= x_{2^n} \quad x_{2^n\ 1} = \ {}_1$$

- Recall that ${}_j$ are singular values of $A + B$
- Result for smallest eigenvalue value of Wishhart matrices by Edelman
- Note that is very large compared to mean spacing (O(1/n) instead of $2\ ^{n}$)

The relation with iid Bernoullis

Let x_j be the eigenvalues of H. Then

$$x_j = \frac{1}{2}\sum_{k\in S_j} {}_k \quad \frac{1}{2}\sum_{k\in S_j^c} {}_k$$

for some $S_j \quad \{1,\ldots,n\}$.

Then

$$d \ {}_n = \frac{1}{2^n}\sum_{j=1}^{2^n} {}_{x_j} = \text{prob. meas. of} \sum_{j=1}^{n} {}_j(B_j \quad 1/2)$$

where B_j are n independent Bernoulli random variables.

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Details

1. Lindenberg condition states:
   - variances σ_k are finite
   - $s_n^2 = \sum_{k=1}^n \sigma_k^2$
   - $\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^n \mathrm{E}\left[(X_k)^2 \cdot \mathbf{1}_{\{|X_k| > \varepsilon s_n\}}\right] = 0$
   - yields convergence to a Normal distribution with variance s_n for sequences of σ_j so that the maximum $< \sqrt[4]{n}$
   - will show that the condition on the max is satisfied with P $\to$ 1 as $n \to \infty$
   - a Berry-Esseen estimate yields an error of $1/\sqrt{n}$

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   - will show that the condition on the max is satis ed with P $!$ 1 as $n\;!\;1$
   - a Berry-Esseen estimate yields an error of $1{=}\sqrt{n}$
2. For the computation of the Fourier transform :
   1. Fourier transform of $\frac{1}{\sqrt{n}}\; _j(B_j \quad 1{=}2)$ is cos $\frac{t\,_j}{2\sqrt{n}}$
   2. Fourier transform of the DoS is then $\prod_j$ cos $\frac{t\,_j}{2\sqrt{n}}$

Random Matrix Theory

Have to show that $_n$ $\sqrt{n}$ when ^{2} of matrix entries is $1/N$

Our Numerics

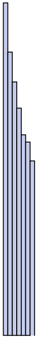

Figure: Spacing distribution for the unfolded spectrum.

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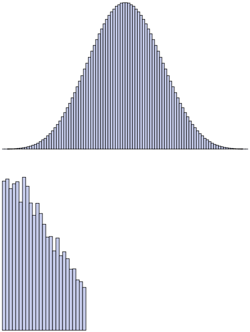

Figure: Density of states and ground state energy gap distribution for Gaussian quadratic form of Fermi operator. Here $n = 16$ (for a sample size of about 50).

Future study

Further questions we want to examine:

- Rate of convergence can probably be improved.
- The bottom eigenvalue of a band covariance matrix.
- In the bulk, the eigenvalues appear to form a Poisson process on the line.
- Speculation: relation to the Berry-Tabor conjecture. Generic integrable system $\Rightarrow$ Poisson statistics

Thank you!